4

Annuities

Structure

4.1. Introduction. In this chapter, we discuss about various types of Types of annuities, their present value and amount of an annuity, including the case of continuous compounding.

4.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Annuity.

(ii) Present value of Annuity.

(iii) Deferred Annuity.

4.1.2. Keywords. Annuity, Present Value, Deferred Annuity.

4.2. Annuity.

Annuity is a series of equal payment made over equal interval of time periods.

For example, if a person deposits Rs. 2000 on first of every month for 2 years, it is an annuity. In this annuity amount of Rs. 2000 paid every month is called **instalment** of the annuity. Because the time difference between two instalments is one month, so the **payment** period of this annuity is one month. Besides this, since the time period between first and last payments is two years i.e. 24 months, so the **term** of the annuity is 24 months.

In annuity certain, number and amount of instalments is fixed and there is no change in then be causes of any contingency. For example instalment paid in recurring deposit in a bank, and for purchase of a plot of land are Annuities Certain.

In annuity contingent, instalments are paid till the happening of some specified event. For example, premium on an insurance policy is paid only as long as the policy holder is alive. In case of his/her then death before the maturity of the policy, further instalments are not paid.

In annuity perpetual, there is no time limit for payment of instalments, they are paid for ever. For example, the instalments of interest earned by endowment fund is a perpetual annuity as they are received regularly forever.

Besides this, if the payment of the instalments is made at the beginning of the corresponding period it is called a Annuity Due and if made at the end of the period, it is called Annuity Immediate. Annuity immediate is called ordinary annuity also. The total amount, to be received, after the maturity of the annuity is the sum of the accumulated values (principal + interest) of all the instalments paid.

Case I. When the annuity is annuity immediate

Let a and n be the amount and number of instalments of an annuity immediate. Further let r be the rate of interest per period. Since 1st instalment is paid at the end of first period, so it will earn on interest for (n-1) periods. Similarly 2nd instalment will earn interest for (n-2) periods and so an. Second last instalment will earn interest for 1 period only and last instalment will not earn any interest.

So total amount of the annuity

$$= a\left(1+\frac{r}{100}\right)^{n-1} + a\left(1+\frac{r}{100}\right)^{n-2} + \cdots + a\left(1+\frac{r}{100}\right) + a$$

$$= a\left[\left(1+\frac{r}{100}\right)^{n-1} + \left(1+\frac{r}{100}\right)^{n-2} + \cdots + \left(1+\frac{r}{100}\right) + 1\right]$$

$$= a[(1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1]$$

$$= a[1 + (1+i) + \cdots + (1+i)^{n-2} + (1+i)^{n-1}]$$

Now this is a geometrical progression and so the sum is given by

$$\text{Amount} = a\left[\frac{1((1+i)^n-1)}{1+i-1}\right] = 0 \qquad \left\{since,\ \ S = \left[a\,\frac{(r^n-1)}{r-1}\right]\right\}$$

$$= a\left[\frac{1((1+i)^n-1)}{i}\right].$$

Case 2. When the annuity is annuity due.

In this case, first instalment is paid at the beginning of 1st period, 2nd instalment at the beginning of 2nd period and so on. So 1st instalment will earn interest for n period, 2nd instalment for (n-1) periods and so on. Last instalment will earn interest for one period only.

$$\text{So Amount} = a\left(1+\frac{r}{100}\right)^n + a\left(1+\frac{r}{100}\right)^{n-1} + \cdots + a\left(1+\frac{r}{100}\right)$$

$$= a[(1+i)^n + (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i)]$$

$$= a[(1+i) + \cdots + (1+i)^{n-2} + (1+i)^{n-1} + (1+i)^n]$$

$$= a\left[\frac{(1+i)((1+i)^n-1)}{1+i-1}\right]$$

$$= a\frac{(1+i)}{i}[(1+i)^n - 1]$$

$$\text{or Amount} = a\left[\frac{(1+i)^n-1}{\frac{i}{1+i}}\right].$$

4.2.1. Example. A person deposits Rs. 2000 per month in a bank for 2 years. If bank pays compound interest at the rate of 8% p.a. find the amount he will receive if the annuity is (i) immediate and (ii) due.

Solution. $a = Rs. 2000, n = 12 \times 2 = 24$ months, $r = \frac{8}{12} = \frac{2}{3}\%$ monthly

(i) Annuity immediate

$$A = a\left[\frac{(1+i)^n-1}{i}\right], \text{ where } i = \frac{2}{3} \times \frac{1}{100} = \frac{2}{300}$$

$$\therefore \quad A = 2000\frac{\left[\left(1+\frac{2}{300}\right)^{24} - 1\right]}{\frac{2}{300}}$$

$$= \frac{2000\times300}{2}[1.1729 - 1] = 1000 \times 300 \times 0.1729$$

$$= Rs. 51870$$

(ii) Annuity due

$$A = a\left[\frac{(1+i)^n-1}{\frac{i}{1+i}}\right]$$

$$= 2000\frac{\left[\left(1+\frac{2}{300}\right)^{24}-1\right]}{\frac{2/300}{1+2/300}}$$

$$= \frac{2000\times(1.1729-1)}{\frac{2}{300}\times\frac{300}{302}} = 2000 \times 0.1729 \times \frac{302}{2}$$

$$= Rs. 52215.80$$

4.2.2. Example. Find the future value of an ordinary annuity of Rs. 4000 per year for 3 years 10% compound interest rate per annum.

Solution. Here $a = 4000, n = 3, i = \frac{10}{100} = 0.1$

So, $$A = 4000\left[\frac{(1+0.1)^3-1}{0.1}\right]$$

$$= \frac{4000}{0.1}(1.331 - 1) = 4000 \times 0.331 = Rs. 13240$$

4.2.3. Example. Find the future value of an annuity due of Rs. 5000 per year for 10 years at rate of 12 % p.a. the interest being compounded half yearly.

Solution. Here $a = 5000$, $n = 10 \times 2 = 20$ half years, $r = \frac{12}{2} = 6\%$ half yearly.

So $$i = \frac{r}{100} = \frac{6}{100} = 0.06$$

Now $$\text{Amount} = 5000\left[\frac{(1+0.06)^{20}-1}{\frac{0.06}{1+0.06}}\right]$$

$$= 5000[(1.06)^{20} - 1] \times \frac{1.06}{0.06}$$

Let $x = (1.06)^{20}$

Therefore, $\log x = 20\, log 1.06$

$$= 20 \times 0.0253 = 0.5061$$

Therefore, $x = AL[0.5061] = 3.2071$

Now $$\text{Amount} = 5000(3.2071 - 1) \times \frac{1.06}{0.06}$$

$$= 5000 \times 2.2071 \times \frac{1.06}{0.06} = Rs. 194960.5$$

4.2.4. Example. Find the future amount of Rs. 50000 payable at the end of each quarter for 5 years at 10 % p.a. compounded quarterly.

Solution. $a = 40000$, $n = 5 \times 4 = 20$ quarters, $r = \frac{10}{4} = 2.5\%$ quarterly.

So $$i = \frac{2.5}{100} = 0.025$$

Now $$\text{Amount} = a\left[\frac{(1+i)^n-1}{i}\right]$$

$$= 5000\left[\frac{(1+0.025)^{20}-1}{0.025}\right]$$

Let $(1.025)^{20} = x$

Therefore, $\log x = 20\log(1.025) = 20 \times 0.0107 = 0.2145$

So $x = AL[0.2145] = 1.6386$

Thus $$\text{Amount} = 50000 \times \left(\frac{1.6386-1}{0.025}\right)$$

$$= 50000 \times \frac{0.6386}{0.025} = \text{Rs. } 127720$$

To find the instalment of given annuity when amount is given

4.2.5. Example. What instalment has a person to pay at the end of each year if he wants to get Rs. 5,00,000 after 10 years at 5% compound rate of interest per annum.

Solution. We know that

$$\text{Amount, } A = a\left[\frac{(1+i)^n-1}{i}\right]$$

Given $A = Rs. 500000, n = 10, r = 5$ so $i = 0.05$

Now $$500000 = a\left[\frac{(1+0.05)^5-1}{0.05}\right]$$

$$= a\left[\frac{(1.05)^{20}-1}{0.05}\right]$$

$$= a\left[\frac{1.6289-1}{0.05}\right] = \frac{0.6289}{0.05}a$$

Therefore, $$a = \frac{500000\times0.05}{0.6289} = Rs. 39751.95$$

4.2.6. Example. A company creates a sinking fund to provide for paying Rs. 1000000 debt maturing in 5 years. Find the amount of annual deposits at the end of each year if rate of interest is 18% compounded annually.

Solution. $A = 1000000, n = 5, r = 18\%$ so $i = \frac{18}{100} = 0.18$

$$A = a\left[\frac{(1+i)^n-1}{i}\right]$$

$$1000000 = a\left[\frac{(1+0.18)^5-1}{0.18}\right]$$

$$= a\left[\frac{(1.18)^5-1}{0.18}\right]$$

Let $x = (1.18)^5$

So $\log x = 5log1.18 = 5 \times 0.07191 = 0.3595$

$x = AL[0.3595] = 2.2877$

$$1000000 = a\left[\frac{2.5877-1}{0.18}\right]$$

or $$a = \frac{1000000\times0.18}{1.2877} = Rs. 151553.42$$

4.2.7. Example. A machine costs Rs. 1,50,000 and has a life of 10 years. If the scrap value of the machine is Rs. 5000, how much amount should be accumulated at the end of each year so that after 12 years a new machine could be purchased after 10 years at the same price. Annual compound rate of interest is 8 %.

Solution. Amount required after 10 years = $150000 - 5000 = 145000$

we are given $A = 145000, n = 12, r = 8\%, i = .08$

$$A = a\left[\frac{(1+i)^n-1}{i}\right]$$

$$145000 = a\left[\frac{(1+0.08)^{10}-1}{0.08}\right]$$

$$= a\left[\frac{2.1589-1}{0.08}\right]$$

Therefore, $a = \frac{145000\times0.08}{1.1589} = Rs.10009.49$

4.2.8. Example. Find the minimum number of years for which an annuity of Rs. 2000 must sum in order to have at least total amount of Rs. 32000 at 5% compound rate of interest

Solution. $A = 32000, a = 2000, r = 5\%, i = .05$

$$32000 = 2000\left[\frac{(1+0.05)^n-1}{0.05}\right]$$

or $\frac{32000\times0.05}{2000} = (1.05)^n - 1$

or $(1.05)^n = 1.8$

Taking logarithms of both sides

$$n.\log(1.05) = log1.8$$

$$n \times 0.0212 = 0.2553$$

$$n = \frac{0.2553}{0.0212} = 12.04$$

Therefore, the amount of annuity will take 13 years to exceed Rs. 32000 as total amount.

4.2.9. Example. What will be the instalment of an annuity having a total amount of Rs. 75000 for 12 years at 8% p.a., rate of interest compounded half yearly.

Solution. We are given that

$$A = 75000,\ n = 12\times2 = 24,\ r = \frac{8}{2} = 4\% \text{ and } i = 0.04$$

So $75000 = a\left[\frac{(1+0.04)^{24}-1}{0.04}\right]$

or $75000\times0.04 = a[(1.04)^{24}-1]$

Let $x = (1.04)^{24}$

$$logx = 24log1.04$$

$$= 24\times0.01703 = 0.4088$$

$$x = AL[0.4088] = 2.5633$$

So $75000\times0.04 = a(2.5633-1)$

$$a = \frac{3000}{1.5633} = Rs.1919.02$$

Amount of an annuity when the interest is compounded continuously

In this case, the amount of the annuity is calculated by using the formula

$$A = a\int_0^n e^{it}dt \quad \text{where } i = \frac{r}{100}$$

4.2.10. Example. In an annuity, Rs. 5000 are deposited each year for 8 years. Find the amount if interest rate of 10% is compounded continuously.

Solution. We are given

$$a = 5000,\ n = 8,\ r = 10\ \% \text{ and } i = \frac{10}{100} = 0.10$$

So $A = 5000\int_0^8 e^{0.1t}dt$

$= 5000\left[\frac{e^{0.1t}}{0.1}\right]_0^8$

$= \frac{5000}{0.1}[e^{0.8} - 1]$

$= 50000(2.71828^{0.8} - 1)$

Let $x = (2.71828)^{0.8}$

$logx = 0.8\ log2.71828$

$= 0.8 \times 04343 = 0.3474$

$x = AL[0.3474] = 2.2255$

So $A = 50000 \times (2.2255 - 1) = Rs. 61275$

4.2.11. Example. A person wants to have Rs. 20000 in his recurring account at the end of 6 years. How much amount he should deposit each year if the rate of interest is 8 % p.a. compound continuously.

Solution. Here $A = 20000,\ n = 6,\ r = 8$ and $i = \frac{8}{100} = 0.08$

Now $A = a\int_0^n e^{it}dt$

$20000 = a\int_0^6 e^{0.08t}dt$

$= a\left[\frac{e^{0.08t}}{0.08}\right]_0^6$

$= \frac{a}{0.08}[e^{0.48} - 1]$

Let $x = e^{0.48}$

$logx = 0.48\ log2 = 0.48 \times 0.4343 = 0.20846$

$x = AL[0.20486] = 1.6161$

So $2000 \times 0.08 = a[1.6161 - 1]$

$a = \frac{1600}{0.6161} = Rs. 2567$

4.3. Present value of an annuity.

Present value of an annuity is equal to the total worth, at the time of beginning of the annuity, of all the future payments that are to be received. This value is equal to the sum of present values of all the instalments.

Let a be the amount of each instalment, n be the term (time periods) of the annuity and r % be the rate of interest per period. Further let V_1, V_2 … V_n be the present values of instalments paid in periods 1,2,…n respectively.

From our previous discussion, we know that the future value of (FV) of an annuity is given by

$$FV = a\left(1+\frac{r}{100}\right)^n$$

So if an instalment is paid at the end of period 1.

Then $$FV = a\left(1+\frac{r}{100}\right)^{n-1}$$

If we want to calculate present value (PV) of this instalment then it is calculated as

$$PV = \frac{a}{\left(1+\frac{r}{100}\right)}$$

So present value of an instalment is the amount of money today which is equivalent to the amount of that instalment, to be received after a specific period. In general, if an instalment, a, is paid in nth period and rate of interest is r % then

$$PV_n = \frac{a}{\left(1+\frac{r}{100}\right)^n} \quad or \quad \frac{a}{(1+i)^n}$$

Now we will find present value of both ordinary annuity and annuity due.

4.4. Ordinary Annuity or Annuity Immediate.

Present value of the annuity

$$V = V_1 + V_2 + V_3 + \cdots + V_n$$

$$= \frac{a}{1+i} + \frac{a}{(1+i)^2} + \frac{a}{(1+i)^3} + \cdots + \frac{a}{(1+i)^n}$$

$$= a\left[\frac{1}{1+i} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \cdots + \frac{1}{(1+i)^n}\right]$$

$$= a\left[\left(\frac{1}{1+i}\right)\left(\frac{1-\left(\frac{1}{1+i}\right)^n}{1-\frac{1}{1+i}}\right)\right]$$

$$= a\left[\left(\frac{1}{1+i}\right)\left(\frac{(1+i)^n-1}{(1+i)^n}\right)\times\left(\frac{1+i}{1+i-1}\right)\right]$$

$$= a\left[\frac{(1+i)^n-1}{(1+i)^n}\times\frac{1}{i}\right] = a\left[\frac{1-(1+i)^{-n}}{i}\right]$$

(ii) Annuity Due

In this type of annuity, each instalment is paid at the beginning of every period. So first instalment is paid at time zero, 2nd instalment at time 1 and so on last instalment is paid at period

(n-1) Hence PV of 1st instalment is equal to a, of 2nd instalment is $\frac{a}{1+\frac{r}{100}}$ of 3rd instalment is $\frac{a}{\left(1+\frac{r}{100}\right)^2}$ and PV of last instalment is $\frac{a}{\left(1+\frac{r}{100}\right)^{n-1}}$.

So $V = V_1 + V_2 + V_3 + \cdots + V_n$

$$= a + \frac{a}{1+\frac{r}{100}} + \frac{a}{\left(1+\frac{r}{100}\right)^2} + \cdots + \frac{a}{\left(1+\frac{r}{100}\right)^{n-1}}$$

$$= a\left[1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-1}}\right]$$

$$= a\left[\frac{1-\left(\frac{1}{1+i}\right)^n}{1-\frac{1}{1+i}}\right]$$

$$= a(1+i)\left[\frac{1-(1+i)^{-n}}{\frac{1+i-1}{1+i}}\right]$$

$$= a(1+i)\left[\frac{1-(1+i)^{-n}}{i}\right] = a\left[\frac{1-(1+i)^{-n}}{\frac{i}{1+i}}\right].$$

4.4.1. Example. Find the present value of an ordinary annuity of Rs. 1500 per year for 5 years at 8% rate of interest.

Solution. $V = a\left[\frac{1-(1+i)^{-n}}{\frac{i}{1+i}}\right]$

Here $a = 1500,\ n = 5,\ r = 8$ and thus $i = \frac{8}{100} = 0.08$

So $V = 1500\left[\frac{1-(1+0.08)^{-5}}{0.08}\right]$

$$= \frac{1500}{0.08}[1-(1.08)^{-5}]$$

Let $x = (1.08)^{-5}$

$$log x = -5\, log 1.08$$

$$= -5 \times 0.0334$$

$$= (-0.1670 + 1) - 1$$

[Since, mantissa can never be negative so making the value positive, we add and substract 1 to the negative value]

$$= \bar{1}.8330$$

So $V = \frac{1500}{0.08}[1 - 0.6806]$

$= \frac{1500 \times 100}{8} \times 0.3194 = Rs. 5988.75$

4.4.2. Example. Find the present value of an annuity due of Rs. 800 per year for 10 years at a rate of 4 % p.a.

Solution. We are given

$a = 800,\ n = 10,\ r = 4$ % and so $i = \frac{4}{100} = 0.04$

Now $V = a\left[\frac{1-(1+i)^{-n}}{(1+i)}\right] = a[1 - (1+i)^{-n}]\left(\frac{1+i}{i}\right)$

$= 800[1 - (1 + 0.04)^{-10})]\left(\frac{1.04}{0.04}\right)$

Let $x = (1.04)^{-10}$

$\log x = -10 \log(1.04)$

$= -10 \times 0.01703$

$= -0.1703 + 1 - 1 = \bar{1}.8297$

$x = AL[\bar{1}.8297] = 0.6756$

So now $V = 800[1 - 0.6756]\left(\frac{1.04}{0.04}\right)$

$= \frac{800 \times 0.3244 \times 1.04}{0.04} = Rs. 6747.52$

4.4.3. Example. A dealer sells a scooter to a customer on the condition that he will pay Rs 10000 in cash and balance to be paid in 36 month end instalment of Rs 400. If rate of interest is 12 % p.a. find the cash price of the scooter.

Solution. We are given

$a = 400,\ n = 36,\ r = 1$ % per month and so $i = \frac{1}{100} = 0.01$

Now $V = a\left[\frac{1-(1+i)^{-n}}{i}\right]$

$= 400\left[\frac{1-(1+0.01)^{-36}}{0.01}\right]$

$= \frac{400}{0.01}[1 - (1.01)^{-36}]$

Let $x = (1.01)^{-36}$

$\log x = -36 \log(1.01)$

$= -36 \times 0.00432$

$= -0.15557 + 1 - 1 = \bar{1}.84443$

$x = AL[\bar{1}.84443] = 0.6989$

So now $V = 40000[1 - 0.6989]$

$= 40000 \times 0.3011 = Rs. 12044$

PV of 36 instalments = Rs. 12044

Cash paid = Rs. 10000

So cash price of the scooter = 12044 + 10000 = 22044

4.4.4. Example. A person takes a loan from a finance company for construction of a house, to be repayable in 120 monthly instalments of Rs. 1020 each. Find the present value of the instalments if the company charges interest @ 9 % p.a.

Solution. We are given

$$a = 1020,\ n = 120,\ r = \frac{9}{12} = 0.75\% \text{ per month and } i = \frac{0.75}{100} = 0.0075$$

Now $V = a\left[\frac{1-(1+i)^{-n}}{i}\right]$

$$= 1020\left[\frac{1-(1+0.0075)^{-120}}{0.0075}\right]$$

$$= \frac{1020}{0.0075}[1 - (1.0075)^{-120}]$$

Let $x = (1.0075)^{-120}$

$\log x = -120 \log(1.0075)$

$= -120 \times 0.003245$

$= -0.3894 + 1 - 1 = \bar{1}.6106$

$x = AL[\bar{1}.6106] = 0.4079$

Hence $V = \frac{1020}{0.0075}[1 - 0.4079]$

$$= \frac{1020}{0.0075} \times 0.5921 = Rs. 80525.64$$

Type 2. To find amount of instalment when present value is given

4.4.5. Example. Find the amount of instalment on a loan of Rs 40000 to be payable in 10, at the end of year, equal instalments at a rate of 10 % interest per annum.

Solution. We are given

$$V = 40000,\ n = 10,\ r = 10\% \text{ or } i = \frac{10}{100} = 0.1$$

Now $V = a\left[\frac{1-(1+i)^{-n}}{i}\right]$

$$40000 = a\left[\frac{1-(1+0.1)^{-10}}{0.1}\right]$$

or $$a=\frac{40000\times 0.1}{1-(1.1)^{-10}}$$

Let $$x=(1.1)^{-10}$$

$$\log x=-10\log(1.1)$$

$$=-10\times 0.04139$$

$$=-0.4139+1-1=\bar{1}.5861$$

$$x=AL[\bar{1}.5861]=0.3856$$

Hence $$V=\frac{4000}{1-0.3856}$$

$$=\frac{4000}{0.6144}=Rs.6510.42$$

4.4.6. Example. A person takes a loan of Rs. 600000 to be repaid in 60 equal end of month instalments at a rate of 8 % per annum. Find the amount of each instalment.

Solution. We are given

$$V = 600000,\ n = 60,\ r=\frac{8}{12}=\frac{2}{3}\%\text{ or } i=\frac{2}{300}=\frac{1}{150}$$

Now $$V= a\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$600000= a\left[\frac{1-\left(1+\frac{1}{150}\right)^{-60}}{\frac{1}{150}}\right]$$

or $$600000\times\frac{1}{150}= a\left[1-\left(\frac{151}{150}\right)^{-60}\right]$$

Let $$x=\left(\frac{151}{150}\right)^{-60}$$

$$\log x=-60[\log(151)-\log(150)]$$

$$=-60(2.1790-2.17611)$$

$$=-60\times 0.00289$$

$$=-0.17314+1-1=\bar{1}.82686$$

$$x=AL[\bar{1}.82686]=0.6712$$

Hence $$4000=a[1-0.6712]$$

$$=a\times 0.3388$$

$$a=\frac{4000}{0.3388}=Rs.11806.37$$

So monthly instalment is Rs 11806.37

Type 3. Interest is compounded continuously

Present value in this case is given by

$$V = a\int_0^n e^{-it}dt, \quad \text{where } i = \frac{r}{100}$$

4.4.7. Example. Find the present value of an annuity of Rs. 12000 per year for 4 years at a rate of 8 %. The interest is compounded continuously.

Solution. We are given

$$a = 12000,\ n = 4,\ r = 8\,\% \ \text{ or } i = \frac{8}{100} = 0.08$$

Now $$V = a\int_0^n e^{-it}dt$$

$$= a\left[\frac{e^{-it}}{-i}\right]_0^n$$

$$= -\frac{a}{i}\left[e^{-in} - 1\right]$$

So $$V = \frac{1200}{0.08}\left[(2.71828)^{-4\times 0.08} - 1\right]$$

Let $$x = (2.71828)^{-0.32}$$

$$\log x = -0.32\log(2.71828)$$

$$= 0.32 \times 0.43429$$

$$= -0.1390 + 1 - 1$$

$$= \bar{1}.8610$$

$$x = AL[\bar{1}.8610] = 0.7261$$

So $$V = -150000 \times (0.7261 - 1)$$

$$= -150000 \times (-0.2739) = Rs.\,41085$$

4.5. Deferred Annuity.

Deferred annuity is an annuity in which payment of first instalment is made after lapse of some specified number of payment periods. This period is called deferment period. For example, payment of first instalment in case of educational loans and housing loans is paid after a deferment period of one to four years.

4.5.1. Amount of deferred annuity.

Let a be the instalment, n be the time periods, r % be the rate of interest and m the deferment period of a deferred annuity. Amount of this annuity is same as in case of other annuities. This amount is not affected by deferment period.

1. Annuity immediate

$$A = a\left[\frac{(1+i)^n - 1}{i}\right]$$

2. Annuity due

$$A = a\left[\frac{(1+i)^n-1}{\frac{i}{1+i}}\right]$$

4.5.2. Present value of a deferred annuity.

Let $V_1, V_2 \ldots V_n$ be the present values of the 1st, 2nd, nth instalments respectively.

Case 1. Annuity immediate

Since m is the deferment period, so 1st instalment will be paid after (m+1) periods, 2nd after (m+2) periods and the last instalment is paid after (m+n) periods, so

$$V_1 = \frac{a}{\left(1+\frac{r}{100}\right)^{m+1}}, \qquad V_2 = \frac{a}{\left(1+\frac{r}{100}\right)^{m+2}}, \ \ldots \text{ and } \qquad V_n = \frac{a}{\left(1+\frac{r}{100}\right)^{m+n}}$$

$$V_1 = \frac{a}{(1+i)^{m+1}}, \qquad V_2 = \frac{a}{(1+i)^{m+2}} \quad \ldots.. V_n = \frac{a}{(1+i)^{m+n}}$$

Now the present value of the annuity

$$\begin{aligned} V &= V_1 + V_2 + \cdots + V_n \\ &= \frac{a}{(1+i)^{m+1}} + \frac{a}{(1+i)^{m+2}} + \cdots + \frac{a}{(1+i)^{m+n}} \\ &= \frac{a}{(1+i)^m}\left[\frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^n}\right] \\ &= \frac{a}{(1+i)^m}\left[\frac{\frac{1}{1+i}\left(1-\left(\frac{1}{1+i}\right)^n\right)}{1-\frac{1}{1+i}}\right] \\ &= \frac{a}{(1+i)^m}\left[\frac{1-\left(\frac{1}{1+i}\right)^n}{i}\right] \\ &= a(1+i)^{-m}\left[\frac{1-(1+i)^{-n}}{i}\right] \end{aligned}$$

Case II. Annuity Due

In this case, 1st instalment is paid after m periods, 2nd after (m+1) periods and so on. Last instalment will be paid after (m+n-1) periods.

So $$V_1 = \frac{a}{\left(1+\frac{r}{100}\right)^{m}}, \qquad V_2 = \frac{a}{\left(1+\frac{r}{100}\right)^{m+1}}, \ \ldots \text{ and } \qquad V_n = \frac{a}{\left(1+\frac{r}{100}\right)^{m+n-1}}$$

$$V_1 = \frac{a}{(1+i)^{m}}, \quad V_2 = \frac{a}{(1+i)^{m+1}} \quad \ldots.. V_n = \frac{a}{(1+i)^{m+n-1}}$$

Now the present value of the annuity

$$\begin{aligned} V &= V_1 + V_2 + \cdots + V_n \\ &= \frac{a}{(1+i)^{m}} + \frac{a}{(1+i)^{m+1}} + \cdots + \frac{a}{(1+i)^{m+n-1}} \\ &= \frac{a}{(1+i)^m}\left[1 + \frac{1}{1+i} + \frac{1}{(1+i)^2} + \cdots + \frac{1}{(1+i)^{n-1}}\right] \end{aligned}$$

$$= \frac{a}{(1+i)^m}\left[\frac{\left(1-\left(\frac{1}{1+i}\right)^n\right)}{1-\frac{1}{1+i}}\right]$$

$$= a(1+i)^{-m}\left[\frac{1-(1+i)^{-n}}{\frac{i}{1+i}}\right]$$

4.5.3. Example. Find the present value of a deferred annuity of Rs. 8000 per year for 8 years at 10 % p.a. rate, the first instalment to be paid at the end of 4 years.

Solution. We are given

$$a = 8000,\ n = 8,\ m = 4,\ r = 10\,\% \text{ so } i = \frac{10}{100} = 0.1$$

Now

$$V = a(1+i)^{-m}\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$= 8000(1+0.1)^{-4}\left[\frac{1-(1+0.1)^{-8}}{0.1}\right] = 800(1.1)^{-4}\left[\frac{1-(1.1)^{-8}}{0.1}\right]$$

Let $x = (1.1)^{-4}$ and $y = (1.1)^{-8}$

$\log x = -4\log(1.1)$

$= -4 \times 0.04139$

$= -0.1656 + 1 - 1$

$= \bar{1}.8344$

$x = AL[\bar{1}.8344] = 0.6830$

$\log y = -8\log(1.1)$

$= -8 \times 0.04139$

$= -0.3312 + 1 - 1$

$= \bar{1}.6688$

$y = AL[\bar{1}.6688] = 0.4664$

So $$V = 8000 \times 0.6830\left[\frac{1-0.4664}{0.1}\right]$$

$$= 80000 \times 0.6830 \times 0.5336 = Rs.\,29155.90$$

4.5.4. Example. A car is sold for Rs 75000 down and 30 half yearly instalments of Rs. 6000 each, the first to be paid after 4 years. Find the cash price of the car, if rate of interest is 12 % p.a. compounded half yearly.

Solution. We are given

$$a = 6000,\ n = 30,\ m = 3.5 \times 2 = 7,\ r = 6\,\% \text{ so } i = \frac{6}{100} = 0.06$$

Now

$$V = a(1+i)^{-m}\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$= 6000(1+0.06)^{-7}\left[\frac{1-(1+0.06)^{-30}}{0.06}\right] = \frac{6000}{0.06}(1.06)^{-7}[1-(1.06)^{-30}]$$

Let $x = (1.06)^{-7}$ and $y = (1.06)^{-30}$

$\log x = -7\log(1.06)$

$\log y = -30\log(1.06)$

$= -7 \times 0.0253$ $\quad$ $= -30 \times 0.0253$

$= -0.1771 + 1 - 1$ $\quad$ $= -0.7592 + 1 - 1$

$= \bar{1}.8229$ $\quad$ $= \bar{1}.2408$

$x = AL[\bar{1}.8229] = 0.6651$ $\quad$ $y = AL[\bar{1}.2408] = 0.1741$

So $\quad V = 100000 \times 0.6651[1 - 0.174]$

$= 100000 \times 0.6651 \times 0.8259 = Rs. 54930$

Cash price = Cash payment + present value of future instalments

= 75000+54930

= Rs. 129930

4.6. Check Your Progress.

1. A person deposits Rs. 10000 at the end of each year for 5 years. Find the amount, he will receive after 5 years if rate of compound interest is 10% p.a.
2. Calculate the future value of an ordinary annuity of Rs. 8000 per annum for 12 years at 15% p.a. compounded annually.
3. A company has set up a sinking fund account to replace an old machine after 8 years. If deposits in this account Rs. 3000 at the end of each year and rate of compound interest is 5% p.a. find the cost of the machine.
4. To meet the expenses of her daughter a woman deposits Rs. 3000 every six months at rate of 10% per annum. Find the amount she will receive after 18 years.
5. A sinking fund is created by a company for redemption of debentures of Rs. 1000000 at the end of 25 years. How much funds should be provided at the end of each year if rate of interest is 4% compounded annually.
6. The parents of a child have decided to deposit same amount at every six months so that they receive an amount of Rs. 100000 after 10 year. The rate of interest is 5% p.a. compounded half yearly.
7. Which is a better investment - An annuity of Rs. 2000 each year for 10 year at a rate of 12 % compounded annually or an annuity of Rs. 2000 each year for 10 years at a rate of 11.75 % compounded half yearly.
8. Find the present value of an annuity due of Rs 4000 per annum for 10 years at a rate of 8 % per annum.
9. Find the present value of an ordinary annuity of Rs. 5625 per year for 6 year at rate of 9 % per annum.
10. Find the present values of an ordinary annuity of Rs. 5000 per six months for 12 years at rate of 4 % p.a. if the interest is compounded half yearly.
11. John buys a plot for Rs 3,00,000 for which he agrees to equal payments at the end of each year for 10 years . If the rate of interest is 10 % p.a. find the amount of each instalment.
12. Lalita buys a house by paying Rs 1,00,000 in cash immediately and promises to pay the balance amount in 15 equal annual instalments of Rs. 8000 each at 15 % compound interest rate. Find the cash price of the house.

13. Find the amount of instalment on a loan of Rs. 250000 to be paid in 20 equal annual instalments at a rate of 8 % per annum.
14. A persons buys a car for Rs. 2,50,000. He pays Rs. 1,00,000 in cash and promises to pay the balance amount in 10 annual equal instalments. If the rate of interest is 12 % per annum, find the instalment.
15. Find the present value of an annuity of Rs 11000 per year for 6 years at a rate of 11 % if the interest is compounded continuously.
16. Find the present value of a deferred ordinary annuity of 12000 per year for 10 years at a rate of 6 % p.a., the first instalment being paid after 3 years.

Answers.

1. Rs. 61050 2. Rs. 232008 3. Rs. 28647 4. Rs. 143754.48

5. Rs. 24081.9 6. Rs.3924.64

7. Amount 1st = Rs. 35098 and for 2nd = Rs. 34674. So first investment is better.

8. Rs. 28987.2 9. Rs. 25233.75 10. Rs. 94570 11. Rs. 1127.90

12. Rs. 146779 13. Rs. 25463.43 14. Rs. 26548.67 15. Rs. 53141

16. Rs. 74154.35

4.7. Summary. In this chapter, we discussed about annuities and its types and considered some examples to understand the topic.

Books Suggested.

1. Allen, B.G.D, Basic Mathematics, Mcmillan, New Delhi.
2. Volra, N. D., Quantitative Techniques in Management, Tata McGraw Hill, New Delhi.
3. Kapoor, V.K., Business Mathematics, Sultan chand and sons, Delhi.